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BIOGRAPHY.

JAMES JOSEPH SYLVESTER, A. M., LL. D., F. R. S., D. C. L.

BY DR. GEORGE BRUCE HALSTED, AUSTIN, TEXAS.

AN adequate life of James Joseph Sylvester has never been written, and probably never will be while he lives.

The present biography will aim neither at completeness nor even symmetry, since so brief a sketch of so great a man can be of permanent value only by giving what the writer alone knows, or for some particular reason happens to know better than others.

While Sylvester was yet a boy he won a large sum of money offered for the solution of a difficult problem in combinations by a firm who actually wished to use the solution in their business. Partly on the strength of this he went to Cambridge and though not regularly trained for the great tripos examinations, he came out Second Wrangler in the class of which Griffin was first and the celebrated George Green was fourth wrangler.

As he would not sign the thirty-nine articles of the Established Church, he was not allowed to take his degree, nor to stand for a Fellowship, to which his rank in the tripos entitled him. Sylvester always felt bitterly this religious disbarment.

His denunciation of the narrowness, bigotry, and intense selfishness exhibited in these creed tests was a wonderful piece of oratory in his celebrated address at the Johns Hopkins University.

The writer will never forget the emotion and astonishment exhibited by James Russel Lowell while listening to this unexpected climax.

Some of Sylvester's first published work was done about this time on the mathematics of Fresnel's optical theory and was incorporated into a text-book

JAMES JOSEPH SYLVESTER, A. M., LL. D., F.R.S.,D.C.L.

by his class-mate Griffin who had the very highest opinion of Sylvester's originality and power. He was soon appointed to a professorship in London, from which place he was called to America by the University of Virginia.

The cause of his sudden abandonment of the University of Virginia the writer has often heard related by the Rev. Dr. R. L. Dabney as follows: In Sylvester's class were a pair of brothers, stupid and excruciatingly pompous. When Sylvester pointed out one day the blunders made in a recitation by the younger of the pair, this individual felt his honor and family pride aggrieved, and sent word to Professor Sylvester that he must apologise or be chastized.

Sylvester bought a sword-cane, which he was carrying when waylaid by the brothers, the younger armed with a heavy bludgeon.

An intimate friend of Dr. Dabney's happened to be approaching at the moment of the encounter. The younger brother stepped up in front of Professor Sylvester and demanded an instant and humble apology.

Almost immediately he struck at Sylvester, knocking off his hat, and then delivered with his heavy bludgeon a crushing blow directly upon Sylvester's bare head.

Sylvester drew his sword-cane and lunged straight at him, striking him just over the heart. With a despairing howl, the student fell back into his brother's arms screaming out "I am killed"! "He has killed me". Sylvester was urged away from the spot by Dr. Dabney's friend, and without even waiting to collect his books, he left for New York, and took ship back to England.

Meantime a surgeon was summoned to the student, who was lividly pale, bathed in cold sweat, in complete collapse, seemingly dying, whispering his last prayers. The surgeon tore open his vest, cut open his shirt, and at once declared him not in the least injured. The fine point of the sword cane had struck a rib fair, and caught against it, not penetrating.

When assured that the wound was not much more than a mosquito-bite, the dying man arose, adjusted his shirt, buttoned his vest, and walked off, though still trembling from the nervous shock. Sylvester was made head professor of mathematics of the Royal Military Academy at Woolwich, a position which he held until the early period set by the English military laws for conferring the life-pension.

He thus happened to be free to accept a position at the head of mathematics in the Johns Hopkins University at its organization. With British conservatism, he stipulated that his traveling expenses and annual salary of five thousand dollars should be paid him in gold, and this fixed, he came a second time to America.

The fame of his coming preceded him, for by this time he was ranked by Kelland in the Encyclopaedia Britannica as the very foremost living English mathematician. The only possible sharer of this proud preeminence was his life-long friend Cayley.

Just at this time there was held in New York an Intercollegiate Mathematical Contest open to all the colleges of America, with Peter S. Michie of

the Army and Simon Newcomb of the Navy as examiners, in which the writer representing Princeton, won a prize of two hundred dollars, and Thomas Craig of Lafayette received honorable mention. Perhaps largely on the strength of this, we were both appointed among the first twenty fellows at the organization of the Johns Hopkins University. The writer having an intense desire to study Sylvester's own creations with him, became alone his first class in the new University. Sylvester gives in his celebrated address a graphic account of the formation of that first-class as illustrating the mutual stimulus of student and professor.

The text-book was Salmon's Modern Higher Algebra, dedicated to Sylvester and Cayley as made up chiefly from their original work.

The professor broke every rule and canon of the Normal Schools and Pedagogy, yet was the most inspiring teacher conceivable. Every thing, from music to Hegel's metaphysics, linked into the theory of Invariants, combined with the precious personal data, and charming unpublished reminiscences of all the great mathematicians of the preceding generation.

Such a course in the creation of modern mathematics, with most precious, elsewhere unattainable, historic indications, will perhaps never be paralleled. It went on not only at the appointed hours, but the professor would send for his class late at night, while at other times they took excursions together to Washington. The incidents of those two formative years spent by the writer in most intimate association with one of the great historic personages of science can never be forgotten. It was during this period that Sylvester founded the American Journal of Mathematics, and it is due to his particular wish that it was given the quarto form.

Then began a new productive period in his life, the astounding activity and marvelous results of which can be faintly estimated by consulting the pages upon pages taken up in the Johns Hopkins Bibliographia Mathematica merely to enumerate the title of the memoirs and papers produced. The entire space devoted to this sketch would be inadequate even to begin any critical estimate of his work. The very complete and profound historic and bibliographic account of the theory of Invariants given by Meyer in the Berichte of the deutche mathematische Gesellschaft indicates very fairly Sylvester's final place in the history of that all-pervading subject. His original contributions to many other parts of the vast structure of modern pure analysis are of nearly as great weight.

But throughout his whole life and work Sylvester is an algebraist, an analyst, as distinguished from a geometer; in this respect contrasting sharply with his friend and admirer Professor Clifford, who conceived of everything in terms of space, and was even able to work synthetically in non-Euclidean space.

Professor Sylvester speaks and writes French with perfect freedom, and it is characteristic that certain of his memoirs prepared for German Journals were written in French.

His personal character is of the very highest, without blemish; and of

his aimiability the writer has experienced repeated proofs. No one now doubts that his second advent on this continent begins the present period of mathematical awakening in America, and that he contributed more to the present upheavel of pure science here than all preceding forces combined. Yet some of his pupils and colleagues have declared that he was a poor teacher; and it is possibly true that he would have made a poor *school-teacher*.

But as President Stanley Hall's article in the Forum so well sets forth, no teaching for a real university can be ranked high which is not vitalized by abundant original creative work.

Sylvester himself was so completely of this opinion as to assert that if a mature man who had produced little or no original work applied for a professorship in a university, he must be either a contemptible ignoramus or a selfish hypocrite and an enemy to mankind.

Moreover he maintained that it was the plain duty of any mature man holding a professorship in a real university to resign at once if he had not already been copiously and creatively productive.

He believed that without unceasing original research and published original work there could be no real university teaching, and that any university professor who, without such a basis, pretended to be a good teacher, was, consciously or unconsciously, a selfish fraud.

Sylvester never married, and what he particularly missed in America was the Athenaeum Club. He has told the writer he would rather give up the Royal Society than the Athenaeum Club. When on the death of his friend, the justly admired Henry J. Stephen Smith, the head professorship of mathematics in the University of Oxford, the Savilian Professorship of Geometry, became vacant, Sylvester hastened to apply for it, and no offers could retain him in America after his election to Oxford was announced.

That he still retains his affection for some of his American pupils and follows their work with interest, is hinted by his proposing the writers name to the London Mathematical Society, and by continued assurances of personal regard. Every American who now goes to Oxford should visit Newcollege, and try to catch at least a glimpse of this one of the Immortals, one who has done so much for America.

THE INSCRIPTION OF REGULAR POLYGONS.

By LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

CHAPTER I.

From the time of the early Greeks, geometers have taken a decided interest in the inscription of regular polygons. However, practically nothing was added to the subject as left us by Euclid for twenty centuries—until Gauss published his immortal researches thereon. Up to Gauss' time geometers felt certain that the only regular polygons that could be inscribed geometrically were those having 2^x, 3.2^x, 5.2^x, and 15.2^x sides.

Gauss proved that every regular polygon having for the number of its sides a prime number of the form 2^x+1, or the product of any number of different primes of that form, or a power of 2 times such a number, is geometrically inscriptible. Thus he added to Euclid's list those of 17, 34, 51, 257, etc., sides.

Further, he gave a method of deducing the simplest equations upon whose solution depends the inscription of any particular regular polygon. He made the inscription of the regular n-gon depend upon the solution of the binomial equation $x^n-1=0$, *i. e.*, upon finding the nth roots of unity. When n is odd, the only real root is $+1$. Dividing out the factor $x-1$, we obtain the reciprocal equation $x^{n-1}+x^{n-2}+\ldots\ldots+x+1=0$, considered by Gauss in his *Disquisitiones Arithmeticæ*, by Bachmann in his *Kreistheilung*, and others.

My method of treatment is based upon pure geometric principles. The unknowns in my treatment are all real; in Gauss', all imaginary. Further, the number of the unknowns by my method is just half as great as the number by Gauss'.

It is necessary to consider here only regular polygons of an odd number of sides, those of an even number of sides being derived from them by continued bisection of the angles at the centre.

I will now consider the method here employed to obtain the equations whose solution will give the inscription of the regular polygons. I will first treat a couple of special cases and then pass to the general one.

To form the equation upon which depends the inscription of the regular polygon of 7 sides.

Let $7a=\pi$. Then $\sin 3a=\sin 4a$.

$$\therefore \sin 2a \cos a+\cos 2a \sin a=2 \sin 2a \cos 2a.$$

$$2 \sin a \cos^2 a+\cos 2a \sin a=4 \sin a \cos a \cos 2a.$$

Dividing out $\sin a$, $\quad 2 \cos^2 a+\cos 2a=4 \cos a \cos 2a.$

$$\therefore 2 \cos^2 a+2 \cos^2 a-1=4 \cos a(2 \cos^2 a-1)$$

$$\text{or,}\quad 8 \cos^3 a-4 \cos^2 a-4 \cos a+1=0.$$

Let $x=2 \cos a$, then $\quad x^3-x^2-2x+1=0 \ldots. (1).$

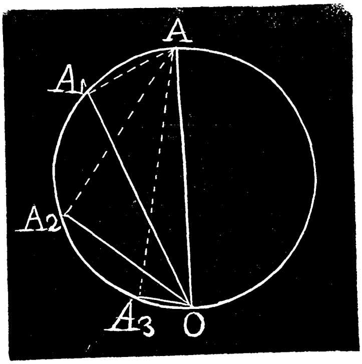
Beginning anew, let $7a'=2\pi$.

Then $\sin 3a' = -\sin 4a'$. Expanding as above, $8\cos^3 a' + 4\cos^2 a' - 4\cos a' - 1 = 0$, which may be written $(-2\cos a')^3 - (-2\cos a')^2 - 2(-2\cos a') + 1 = 0$.

Thus $-2\cos a'$ is a root of equation (1).

Finally, let $7a'' = 3\pi$. Then $\sin 3a'' = \sin 4a''$.

Hence, $8\cos^3 a'' - 4\cos^2 a'' - 4\cos a'' + 1 = 0$. Thus $2\cos a''$ is the third root of (1). These roots may evidently be written thus: $2\cos\frac{\pi}{7}$, $-2\cos\frac{2\pi}{7}$, $2\cos\frac{3\pi}{7}$.



Suppose a circle of unit radius divided at A, A_1, A_2, A_3, etc., into 7 equal parts, and the diameter AO drawn. Join the points A and O to A_1, A_2, A_3.

Then $\angle AOA_1 = \frac{\pi}{7}$; $\angle AOA_2 = \frac{2\pi}{7}$; $\angle AOA_3 = \frac{3\pi}{7}$.

$\therefore\ 2\cos\frac{\pi}{7} = 2\cos AOA_1 = OA_1$; $2\cos\frac{2\pi}{7} = OA_2$; $2\cos\frac{3\pi}{7} = AO_3$.

Hence, $OA_1, -OA_2$, and OA_3 are the roots of the cubic

$$x^3 - x^2 - 2x + 1 = 0.$$

For a regular polygon of 9 sides, let $9a = \pi$. Then $\sin 4a = \sin 5a$. Expanding, dividing out $\sin a$, and reducing, we find $16\cos^4 a - 8\cos^3 a - 12\cos^2 a + 4\cos a + 1 = 0$. Write $x = 2\cos a$, then $x^4 - x^3 - 3x^2 + 2x + 1 = 0 \ldots.(2)$.

Let $9a' = 2\pi$; then $\sin 4a' = -\sin 5a'$. Expanding we find $16\cos^4 a' + 8\cos^3 a' - 12\cos^2 a' - 4\cos a' + 1 = 0$, which may be written $(-2\cos a')^4 - (-2\cos a')^3 - 3(-2\cos a')^2 + 2(-2\cos a') + 1 = 0$. Hence, $-2\cos a'$ is a root of (2).

Let $9a'' = 3\pi$; then $\sin 4a'' = \sin 5a''$. Lastly, let $9a''' = 4\pi$; then $\sin 4a''' = -\sin 5a'''$. Hence, $2\cos a''$ and $-2\cos a'''$ are roots of equation (2). These four roots may be written $2\cos\frac{\pi}{9}, -2\cos\frac{2\pi}{9}, 2\cos\frac{3\pi}{9}, -2\cos\frac{4\pi}{9}$. These roots are easily shown to be the chords $OA_1, -OA_2, OA_3$, and $-OA_4$ in a circle of unit radius.

Since $2\cos\frac{3\pi}{9} = 1$, one root of (2) is 1, and dividing out the factor $x-1$ we obtain the cubic $x^3 - 3x - 1 = 0$, whose roots are the chords $OA_1, -OA_2, -OA_4$ in the unit circle.

Proceeding in a similar manner, we obtain the equations upon whose solution depends the inscription of other regular polygons:

For 11 sides, $x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0$.

For 13 sides, $x^6-x^5-5x^4+4x^3+6x^2-3x-1=0$.

If we form a table of the coefficients of these equations, we discover a curious resemblance to the famous Triangle of Pascal, each column in the triangle gives two colums in our table. The table up to 19 sides is as follows, N being the number of sides of the regular polygon:

N										
3	1	1								
5	1	1	1							
7	1	1	2	1						
9	1	1	3	2	1					
11	1	1	4	3	3	1				
13	1	1	5	4	6	3	1			
15	1	1	6	5	10	6	4	1		
17	1	1	7	6	15	10	10	4	1	
19	1	1	8	7	21	15	20	10	5	1

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A M., (Princeton); Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the August Number.]

PROPOSITION VI. *The hypothesis of an obtuse angle, if even in a single case it is true, always in every case it alone is true.*

PROOF. Let the join CD (Fig. 5) make obtuse angles with any two equal perpendiculars AC, BD, standing upon any other straight AB.

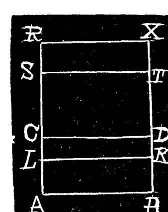


FIG. 5

CD will be (P. III.) less than this AB.

Assume in AC and BD produced any two mutually equal portions CR and DX; and join RX.

Now I investigate the angles at the join RX, which certainly (P. I.) will be mutually equal.

If they are obtuse we have our assertion.

But they are not right, because thus we would have a case for the hypothesis of right angle, which (P. V.) would have no place for the hypothesis of obtuse angle. But neither are they acute

For thus RX would be (P. III.) greater than this AB; and still more therefore greater than CD itself. But that this cannot be is thus shown. If the quadrilateral $CDXR$ is taken to be filled up by straights cutting off from

these CR, DX, portions mutually equal, this implies transition from the sect CD, which is less than AB itself, to RX greater than it, verily transition through a certain ST equal to this AB. But that this is absurd in the present hypothesis follows so; because thus (P. IV.) we have a case for the hypothesis of right angle, which (P. V.) would leave no place for the hypothesis of obtuse angle. Therefore the angles at the join RX must be obtuse.

Then, equal portions AL, BK being assumed in AC, BD; in a similar manner we show the angles at the join LK cannot be acute toward this AB; because thus it would be greater than AB, and still more therefore greater than than the sect CD. But here would be found, as above, a certain intermediate between CD less, and LK greater than this AB; an intermediate, I say, equal to AB itself, which certainly, from what was just now observed, would take away every place for the hypothesis of obtuse angle.

Finally from this very cause the angles at the join LK cannot be right; therefore they will be obtuse.

Therefore with the same base AB, the perpendiculars being increased or diminished at will, the hypothesis of obtuse angle will always persist.

But the same ought to be demonstrated for any assumed base.

Let there be chosen (Fig. 6.) for base any one of the aforesaid perpendiculars, as BX suppose.

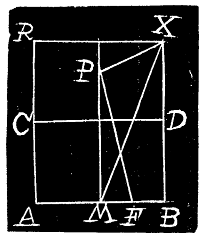


FIG. 6

Let these AB, RX be bisected in the points M and H; and MH joined. MH will be (P. II.) perpendicular to these AB, RX. But the angle at the point B is right by hypothesis; and at the point X obtuse, from what has just now been demonstrated.

Make therefore the right angle BXP toward the parts of this MH. XP will meet MH itself in some point P situated between the points M and H; since on the one hand the angle BXH is obtuse; and, on the other, if XM be joined, the angle BXM (Eu. I. 17.) is acute. Then however, since the quadrilateral $XBMP$ contains three right angles, from what has just now been noted, and one obtuse (Eu. I. 16.) at the point P, because it is external in relation to the internal and opposite right angle at the point H of the triangle PHX; the side XP will be (Cor. I., P. III.) less than the opposite BM. Wherefore, assuming in BM the portion BF equal to this XP, the angles at the join PF will be (P. I.) mutually equal, certainly obtuse, since the angle BFP (Eu. I. 16.) is obtuse because of the right angle interior and opposite FMP. Therefore the hypothesis of obtuse angle abides for any base BX.

But, as above, this hypothesis abides for this base BX, however much the equal perpendiculars are augmented or diminished at will. Therefore it holds, that the hypothesis of obtuse angle, if even in a single case it is true, always in every case it alone is true.

Quod erat demonstrandum.

Translator's Note. This demonstration assumes that in a quadrilateral like $CDXR$ (Fig. 5.) we can make RX differ from CD as little as we choose

by making CR and DX sufficiently small; or that the sect ST in passing from coincidence with CD to coincidence with RX is a continuous variable and so passes through all intermediate values.

PROPOSITION VII. *The hypothesis of acute angle, if even in a single case it is true, always in every case it alone is true.*

This is very easily shown. For if the hypothesis of acute angle should permit any case of either other hypothesis, either of right angle, or of obtuse angle, now (from the two preceding propositions) no place would be left for the hypothesis of acute angle; which is absurd.

Therefore the hypothesis of acute angle, if even in a single case it is true, always in every case it alone is true.

Quod erat demonstrandum.

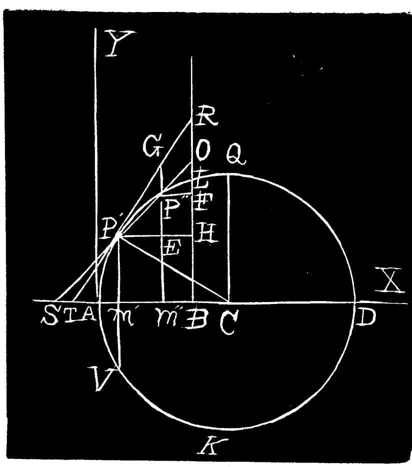
THE FIRST DIFFERENTIAL CO-EFFICIENT OF A CIRCLE.

By Professor JOHN N. LYLE, Ph. D., Professor of Natural Science, Westminster College, Fulton, Missouri.

The equation of the circle AQD, Fig. 1. referred to the Cartesian rectangular co-ordinate axes AY, AX is $y^2=2Rx-x^2$ .... (1).

Let a point move from A in the path of the curve $AQDK$. Further, let the motion be such that wherever possible dx, the rate of variation of the abscissa x, may be uniform.

y is a function of x, for it takes a new value corresponding to each new value assumed by the variable x.



When the moving point has arrived at P' the variable x has reached the individual value x' or Am'; and the function y, the corresponding individual value y' or $P'm'$.

We then have $y'^2=2Rx'-x'^2$ ....(2).

Let dx' or $m'B$ stand for the increment to the variable x in the unit of time after reaching the value x' or Am'.

Since x increases uniformly, dx' represents its rate of increase.

Draw BR perpendicular to the axis of abscissas, intersecting the tangent line $TP'R$ in the point R. Draw $P'H$ perpendicular to BR. The line $P'H$ is equal to the constant line $m'B$, and is, consequently, represented by dx'.

Divide dx or $P'H$ into n equal parts. Then $\frac{dx}{n}$ or $\triangle x = \frac{P'H}{n}$ or $P'E$.

Erect the perpendicular EP'' to the line $P'H$ intersecting the curve in the point P'' whose co-ordinates are x'', y''.

Through the points P' and P'' draw the secant line $SP'P''O$.

When the moving point has reached P'', the function y has reached the new value y'', or $y' + \triangle y'$, corresponding to the new value x'', or $x' + \frac{dx'}{n}$, assumed by the variable x.

The equation of the curve corresponding to the point P'' is $y''^2 = 2Rx'' - x''^2$ or $(y' + \triangle y')^2 = 2R(x' + \triangle x') - (x' + \triangle x')^2$, (3), in which $\triangle x'$ and $\triangle y'$ are the increments of the abscissa x and the ordinate y in one nth of the unit of time after the moving point reached P'.

Subtract (2) from (3).

Then $2y'\triangle y' + \triangle y'^2 = 2R\triangle x' - (2x'\triangle x' + \triangle x'^2)$, (4), and $2y'\triangle y' = 2(R - x')\triangle x' - \triangle x'^2 - \triangle y'^2$ and $\triangle y' = \frac{R - x'}{y'}\triangle x' - \frac{\triangle x'^2 + \triangle y'^2}{2y'}$, ...(5).

Multiply both members by n. Then $n \times \triangle y' = \frac{R - x'}{y'}dx' - \frac{n(\triangle x'^2 + \triangle y'^2)}{2y'}$..(6).

As n increases in value and $\triangle x'$ correspondingly diminishes, the product $n \times \triangle x'$, or dx, remains constant.

As n increases and $\triangle y'$ correspondingly diminishes, the product $n \times \triangle y'$ is seen from inspecting the second member of the (6) to be a variable that approaches the constant $\frac{R - x'}{y'}dx$ as its limit.

The value of the variable product $n \times \triangle y'$ consists of two parts, one of which is constant; and the other of which $\frac{n(\triangle x'^2 + \triangle y'^2)}{2y'}$ is a variable that decreases indefinitely without limit.

The part $\frac{R - x'}{y'}dx'$ is due to the tendency of y to vary when it reached the value y', that is, to the rate of variation of y when it equals y'.

$\frac{\triangle x'^2 + \triangle y'^2}{2y'}$ is due to the increment to the tendency of y to vary during one nth of the unit of time after reaching the value y'. This multiplied by n is $\frac{n(\triangle x'^2 + \triangle y'^2)}{2y'}$.

Let $i = \frac{n \times (\triangle x'^2 + \triangle y'^2)}{2y'}$....(7).

Add (7) to (6) and we shall have $n\times\triangle y'+i=\dfrac{R-x'}{y'}\,dx'\ldots.(8)$.

The second member of equation (8) represents the rate of variation of y corresponding to the value y'.

For $n\times\triangle y'+i$ write dy'.

Then $dy'=\dfrac{R-x'}{y'}\,dx'\ \ldots.(9)$.

Hence, dy' stands for the rate of variation of the variable y when it reaches the value y'.

dy' and dx' are called differentials of y and x corresponding to y' and x'.

The ratio between dy' and dx' is called the first differential co-efficient for the point P'.

Dropping the primes we have as a general expression for the first differential co-efficient $\dfrac{dy}{dx}=\dfrac{R-x}{y}\ \ldots.(10)$.

Remark 1. dx, or $n\times\triangle x'=P'H$ or $n\times P'E$, and $n\times\triangle y'=n\times P''E=HO$.

Remark 2. As n increases without limit, P'' approaches P', $P'E$ and $P''E$ diminishes without limit. $n\times P'E$ remains constant, and $n\times P''E$ or HO is a variable approaching the constant HR.

HO is the geometrical equivalent of $n\times\triangle y'$, and HR its limit is the geometrical equivalent of dy' the symbol representing the differential or rate of variation of the variable y corresponding to the value $P'm'$ or y'.

Remark 3. The limit of the increasing variable HO is obtained by adding these to the difference OR between HO and its limit, the constant HR.

The limit of the increasing variable $n\times\triangle y'$ is obtained by adding these to the difference i between $n\times\triangle y'$ and its limit the constant dy'.

Equation (8) is obtained from (7) and (6) in accordance with the Euclidian axiom—"If equals be added to equals, the wholes are equal."

Remark 4. Equation (6) is $n\times\triangle y'=$

$$\frac{R-x'}{y'}\,dx'-\frac{n(\triangle x'^2+\triangle y'^2)}{2y'}.$$

The geometrical equivalent of $n\times\triangle y'$ is HO.

Join $P'C$. Since $x'=Am'$, $R-x'=Cm'$. Hence the geometrical equivalent of $\dfrac{R-x'}{y'}\,dx'$ is $\dfrac{Cm'}{P'm'}\times P'H$.

Remembering that the triangles $Cm'P'$ and RHP' are similar, the angle $m'C'P'$ being equal to HRP', we have $HR:P'H::Cm':P'm'$.

$$\text{Or } HR = \frac{Cm'}{P'm} \times P'H.$$

Hence HR, also, is the geometrical equivalent of $\frac{R-x'}{y'} dx'$.

Produce $P'm'$ to V and join Vp''.

Then $2y' = P'V$ and $\triangle x'^2 + \triangle y'^2 = P'P''^2$ and $n \times P'P'' = P'O$.

Hence $\frac{n \times (\triangle x'^2 + \triangle y'^2)}{2y'} = \frac{P'P''}{P'V} \times P'O$.

The angle $VP''P'$ is equal to $m'CP'$.

Hence, the triangle $P'OR$ and $VP'P''$ are similar and consequently $P'V : P'P'' :: P'O : OR$.

That is, $OR = \frac{P'P''}{P'V} \times P'O$.

It is evident, therefore, that the difference OR between the variable HO and its limit HR is equal to $\frac{P'P''}{P'V} \times P'O$ the difference between $n \times \triangle y'$ or $n \times P''E$ and its limit dy' or $\frac{Cm'}{P'm'} \times P'H$.

The line OR is the geometrical equivalent of i in equations (7) and (8).

Remark 5. The equality between the quantities i and $\frac{n(\triangle x'^2 + \triangle y'^2}{2y'}$, or between OR and $\frac{P'P''}{P'V} \times P'O$ explains what has been called the compensation of errors in the Leibnitzian method.

Bledsoe in his Philosophy of Mathematics, pages 155 and 156, shows that Bishop Berkeley deserves the credit of discovering and first stating this principle of the compensation of errors.

In Gillespie's translation of Compte's Philosophy of Mathematics, page 100, Compte says—"After various attempts more or less imperfect, a distinguished geometer, Carnot, presented at last the true direct logical explanation of the method of Leibnitz, by showing it to be founded on the principle of the necessary compensation of errors, this being, in fact, the precise and luminous manifestation of what Leibnitz had vaguely and confusedly preceived."

The demand of the Leibnitzian School that the line OR is so diminutive that the lines OH and RH are equal in length discredits Euclid's second axiom and stands as an obstacle in the way of correct theory of a the Calculus.

Remark 6. As n receives the successive finite values 10, 100, 1000, &c., P'' approaches P' and O approaches R.

As P'' approaches P' in accordance with the hypothesis in hand, the lines $P'm'$, $m'C$, CP', $P'H$, HR and $P'R$ remain constant, whilst $P'E$, $P''E$, $P'P''$ and OR simultaneously diminish without limit.

The angle $VP''P'$ is constant in value and equal to $P'RO$, whilst $P'VP''$ diminishes without limit.

The two following things may be stated respecting the series 10, 100, 1000, &c.:

1. Every term admitted into the series is a finite number.
2. No last term is attributed to the series.

The line $P'H$ is a constant. Since every value of n is assumed to be finite, every value of $P'E$, which is one nth of $P'H$, must be finite.

Whilst $P'E$ decreases in value without limit as n increases without limit, it is, nevertheless, *indestructible*.

Since by hypothesis it is a part of the constant, finite line $P'H$.

Furthermore, each of the lines $P'E$, and $P''E$, $P'P''$ and OR has *two* ends, and is, therefore, *finite*.

The hypothesis under which we are working prevents P'' from coinciding with P' and hence prevents the annihilation of $P'E$ and $P''E$.

The increments $P'E$ and $P''E$ are the data from which we logically infer the rate of variation of y for the point P'. As we do not obtain logical conclusions by destroying the premises from which we reason it is not required of us to annihilate the indestructible lines $P'E$, $P''E$, or convert the secant $SP'P''O$ into the tangent $TP'R$.

We may draw a straight line through the point P'. If this line cuts the circle in two points, it is a secant; but if it touches the circle in one point only, it is a tangent.

We may also freely admit that a straight line through the point P' may be revolved about that point under a hypothesis that will permit it to make a complete revolution. But when it touches the circumference it does not cut; and when it cuts the circumference it is not a tangent thereto.

The revolving line must, in fact, cease to be a secant before it can be a tangent to the circle. Hence, the absurdity of assuming that a secant ever becomes a tangent.

According to Euclid's Proposition II. Book III.—"If any two points are taken in the circumference of a circle, the chord which joins them falls within the circle". If the straight line joining the points P' and P'' be extended both ways it will be a secant line, the portion of it within the circle being called a chord. Euclid's theorem holds however near P'' may approach P'. The

facts of the Calculus rightly apprehended and correctly explained are found to be in harmony with the data of Euclid's Elements.

Remark 7. The tangent line to the circle at the point A is perpendicular to the axis AX, and $n \times \triangle y$ corresponding to that point is not a variable product having a constant quantity as a limit, but one that increases indefinitely without limit as n increases without limit and $\triangle y$ correspondingly diminishes.

Is there a differential coefficient for the point A? Let us examine and report. For the point A the product $n \times \triangle y$ is not a constant line and is not a variable having a constant line as a limit. If the rate of variation of y wherever possible is either a constant equal to $n \times \triangle y$ or a constant and the limit of $n \times \triangle y$, then for the point A there is no rate of y. For that point $n \times \triangle y$ is a variable that increases indefinitely without limit. A differential coefficient is a ratio between tworates of variation. If either rate is *absent* for any point of the curve, there is no ratio and hence, no differential coefficient for that point.

The general expression for the differential coefficient of a circle is $\frac{dy}{dx} = \frac{R-x}{y}$.

For the point A this expression reduces to the form $\frac{dy}{dx} = \frac{R}{0}$.

This is an absurd collocation of analytical symbols inasmuch as a quotient is represented as possible in the absence of a divisor. This absurd expression $\frac{R}{0}$ occurs at the point A where the line that touches the curve is perpendicular to the axis of abscissas and may be taken to indicate that fact but not because $\frac{R}{0}$ is an expression for the tangent of 90°.

The line AY that touches the quadrant AQ at A is not terminated by the line drawn through the centre and the point Q.

But does not the line CQ intersect AY "at infinity"? Euclid and Lobatschewesky agree in affirming that there is no intersection at any point whatever.

But is not the tangent of 90° ∞ ?

If the tangent of an arc is a line touching one extremity and terminated by a line drawn through the centre and the other extremity, it is evident that 90° has no tangent under this definition. Furthermore, a straight line having two ends is *finite* and cannot appropriately be represented by the hieroglyphic ∞.

Remark 8. Is there a differential co-efficient at the point Q ?

The tangent line to the circle at the point Q is parallel to the axis AX

and $n\times\triangle y$ corresponding to that point is not a variable product having a constant quantity as a limit, but one that decreases indefinitely without limit as n increases without limit and $\triangle y$ correspondingly diminishes. $n\times\triangle y$ for the point Q is not a constant quantity and is not a variable having a constant quantity as a limit. If either of these things be essential to a differential of y and hence to a differential co-efficient, then there can be no differential of y and no differential co-efficient corresponding to the point Q.

But does not the perpendicular to CQ at the point Q make a zero angle with the axis of abscissas? According to both Euclid and Lobatschewsky the two lines perpendicular to the radius CQ do not meet. If lines that make an angle with each other always meet, the perpendiculars to CQ do not make an angle.

What is a zero angle? Is it the ghost of a departed quantity that the Bishop of Cloyne tells about?

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

25. **Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham, Ohio.**

A company engaged an agent to do business for one month at a salary of \$25, giving him goods amounting to \$57.54 and \$32.17 in cash to start with. The agent bought during the month, goods amounting to \$59.91. At the end of the month the goods on hand amounted to \$31.67, and the amount of sales for the month was \$102.97; what was the balance of account?

Solution by W. F. BRADBURY, A. M., Head-Master, Cambridge Latin School, Cambridge, Massachusetts.

Cash received..................	\$32.17
Cash for sales..................	\$102.97
Cash total.....................	\$135.14
Cash spent....................	\$59.91
Cash on hand.................	\$75.23
Salary of agent.......	\$25.00
Goods to be returned and cash....	\$50.23

It is assumed that the agent paid for the goods he bought and re-

ceived cash for the sales made. If not, the agent turns over the debits and credits and if the accounts can be collected it is all the same.

The company came out as follows:

When the agent began,	Assets
Cash......	\$50.23
Goods.....	\$31.67
Total......	\$81.90

Loss, \$89.71—\$81.90=\$7.81.

Also solved by *Hon. J. H. DRUMMOND.*

26. Proposed by E. S. LOOMIS, A. M., Ph. D., Professor of Mathematics, Baldwin University, Berea, Ohio.

You say, "While treating of the pronunciation of those who minister in public, two other words occur to me which are commonly mangled by our clergy. One of *these* (A) is 'covetous,' and its substantive 'covetousness.' I hope some who read *these lines* will be induced to leave off pronouncing *them* (B) 'covetious' and 'covetiousness.'I can assure *them* (C) that when *they* (D) do thus call *them* (E),one at least of *their* (F) hearers has his appreciation of *their* (G) teaching disturbed."

The problem now is, in how many ways can this above quotation be (read or) understood, by supposing various antecedents to the pronouns as per table.

The pronouns	Nouns to which they apply.	No. of nouns.
(A) these	Words,or clergy.	2
(B) them	Words, clergy, readers, or lines.	4
(C) them	Words, clergy, readers, or lines.	4
(D) they	Words, clergy, readers, or lines.	4
(E) them	Words, clergy, readers, or lines.	4
(F) their	Words, clergy, readers, or lines.	4
(G) their	Words, clergy, readers, lines,or hearers.	5

Solution by Hon. JOSIAH H. DRUMMOND, Portland, Maine.

"These" (A) may apply to 2; hence two readings: to each of these two, "them" (B) gives four readings: so that we have, $2\times4=8$ readings and so on through the paragraph and the whole number of readings will be the continued product of $2\times4\times4\times4\times4\times4\times5=10240$.

[Note.—The above problem with a solution may be found in *Bardeen's Complete Rhetoric*, page 415. ED.]

27. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A and B buy a ship for $S=\$80000$, of which A has the a/bth$=\frac{5}{8}$, and B the c/dth$=\frac{3}{8}$, interest. They sell C the m/nth$=\frac{1}{3}$ interest for $P=\$40000$; and then agree that A should *retain* the p/qth$=\frac{7}{12}$, and B the r/sth$=\frac{1}{12}$, interest. How is the purchase-money received from C to be divided between A and B?

I. Solution by the PROPOSER.

Obviously A sold to C the m/nth of the a/bth part,$=\frac{5}{24}$ part, of the ship; and B sold to C the m/nth of the c/dth part,$=\frac{3}{24}$ part, of the ship. Before entering upon their *agreement* specified in the problem, A's part of the purchase-money received from C would have been (a/b) of $\$P=\$25000$; and B's part would have been (c/d) of $\$P,=\$15000$. In order to possess himself of

the p/qth part, $=\frac{7}{12}$ part, of the ship, A buys of B at the same price C bought from both, the $\left[\frac{p}{q}-\frac{a}{b}\left(1-\frac{m}{n}\right)\right]$ th part, $=\frac{4}{24}$ part, of the ship, for $\left(\frac{n}{m}\right)$ $\left[\frac{p}{q}-\frac{a}{b}\left(1-\frac{m}{n}\right)\right]$ of $\$P$, $=\$20000$. As the condition of affairs now is, A *retains* the p/qth part, $=\frac{7}{12}$ part, of the ship; and, consequently, B *retains* the $\left[\frac{c}{d}-\left\{\frac{mc}{nd}+\left[\frac{p}{q}-\frac{a}{b}\left(1-\frac{m}{n}\right)\right]\right\}\right]$th part, $=\left[\left(1-\frac{m}{n}\right)\left(\frac{a}{b}+\frac{c}{d}\right)-\frac{p}{q}\right]$ th part, $=r/s$th part, $=\frac{1}{12}$ part, of the ship. Hence A's share of the purchase-money received from C is $A=\left\{\frac{a}{b}-\left(\frac{n}{m}\right)\left[\frac{p}{q}-\frac{a}{b}\left(1-\frac{m}{n}\right)\right]\right\}$ of $\$P$,

$$=\left[\frac{n}{m}\left(\frac{a}{b}-\frac{p}{q}\right)\right] \text{ of } \$P, =\$5000;$$

and for B's share of the purchase-money received, we have

$$B=\left\{\frac{c}{d}+\left(\frac{n}{m}\right)\left[\frac{p}{q}-\frac{a}{b}\left(1-\frac{m}{n}\right)\right]\right\} \text{ of } \$P$$

$$=\left[\frac{c}{d}+\frac{np}{mq}-\frac{a}{b}\left(\frac{n}{m}-1\right)\right] \text{of } \$P, =\$35000.$$

Note.—By making $p/q=r/s=\frac{1}{3}$, the numerical values of A and B will change places—see *Packard's Commercial Arithmetic*, p. 260, Prob. 30.

II. Solution by FRANK HORN, Columbia, Missouri, and J. M. LOWELL, Philadelphia, Pennsylvania.

II.
1. $\frac{1}{24}=\frac{5}{8}-\frac{7}{12}=$part of ship which A sold.
2. $\frac{7}{24}=\frac{3}{8}-\frac{1}{12}=$part of ship which B sold.
3. $\frac{8}{24}=\frac{1}{24}+\frac{7}{24}=$part of ship which was sold.
4. $\$40000=$amount received for $\frac{8}{24}$ of the ship.
5. $\therefore$ $\$5000=$amount received for $\frac{1}{24}$ of the ship,
6. and $\$35000=7\times\$5000=$amount received for $\frac{7}{24}$ of the ship.

III. $\therefore$ for $\frac{1}{24}$ of the ship, which A sold he received $5000, and for $\frac{7}{24}$, which B sold he received $35000.

Excellent solutions received from *P. S. BERG*, *G. B. M. ZERR*, and *J. H. DRUMMOND*.

PROBLEMS.

29. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A wine-merchant's *apparent* profit is 25% of his sales which are 10% of costless water. What is his *actual* rate per cent. of profit?

30. A chain 100m long, weighing 14 oz. to the foot, is suspended from points on a level 80m apart. What is the sag, the batter at the ends, and the horizontal tension? [*From Wentworth & Hill's High School Arithmetic.*]

31. Proposed by B. F. FINKEL, Professor of Mathematics in Kidder Institute, Kidder, Missouri.

Between Sing-Sing and Tarry-Town, I met my worthy friend, John Brown,
And seven daughters, riding nags, and every one had seven bags;

In every bag were thirty cats, and every cat had forty rats,
Besides a brood of fifty kittens. All *but* the nags were wearing mittens!
Mittens, kittens—cats, rats—bags, nags—Browns,
How many were met between the towns?

[*From Muttoon's Common Arithmetic.*]

Solutions to these problems should be received on or before December 1st.

ALGEBRA.

Conducted by J. M. COLAW, Monteray, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

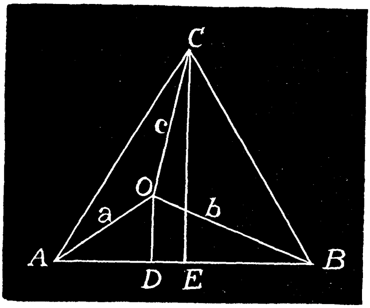
15. Proposed by SETH PRATT, C. E., Assyria, Michigan.

From a point in an equilateral triangle, the distances to the angles are, respectively, 20, 28, and 31 rods. Required a side of the triangle.

II. Solution by G. B. M. ZERR, A. M., Principal of Schools, Staunton, Virginia.

Let the $\angle ACO=\phi$, and $\angle BCO=\theta$. Then, $28^2=31^2+x^2-62x\cos\theta$.

$\therefore \cos\theta=\dfrac{177+x^2}{62x}$.....(1) $20^2=31^2+x^2-62x$



$\cos\phi$. $\therefore \cos\phi=\dfrac{561+x^2}{62x}$....(2) also

$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$

$=\cos 60^\circ=\frac{1}{2}$.

$\therefore \cos\theta\cos\phi-\sqrt{1-\cos^2\theta}\sqrt{1-\cos^2\phi}=\frac{1}{2}$.

$\therefore \cos^2\theta-\cos\theta\cos\phi+\cos^2\phi=\frac{3}{4}$....(3).

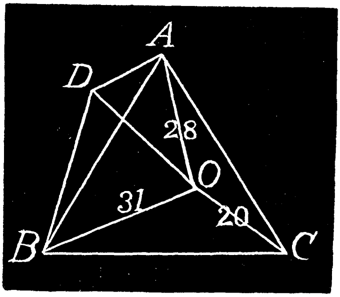
(1) and (2) in (3) gives after reducing $x^4-2145x^2=-246753$.

$\therefore x^2=2023.02785$ or 121.97215.

$\therefore x=44.97+$ or 11.045 according as the point is within or without the triangle.

III. Solution by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

Let ABC be the equilateral triangle, O the point within, $OC=20$, $OB=31$, $OA=28$. On OB construct the equilateral triangle OBD, and join AD.



In $\triangle$s ABD and OBC we have two sides and included angle of one equal to the same in the other. $\therefore AD=OC$.

In $\triangle\ AOD$ the three sides are now given, to find $\angle AOD$. Hence, $\cos AOD=\dfrac{31^2+28^2-20^2}{2\times 31\times 28}$

$=.77494=\cos 39^\circ\ 12'$.

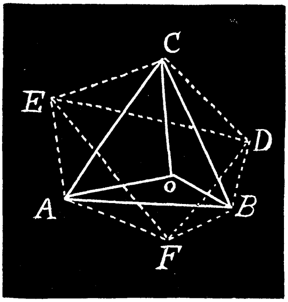
The $\angle AOB=60^\circ+39^\circ\ 12'$. From the two

sides and included angle of OAB we find

$AB=\sqrt{31^2+28^2-2\times31\times28\cos(60^\circ+39^\circ\ 12')}=44.97+$, the side required.

IV. **Solution by J. W. WATSON, Middle Creek, Ohio.**

Let ABC be the given triangle, o the given point. Put $Ao=a=28$, $Bo=b=20$ and $Co=c=31$. Let $S=$ a side of the triangle. Then area$=\frac{1}{4}S^2\sqrt{3}$.



Draw the $\triangle AFB=\triangle AoB$, $\triangle BDC=\triangle BoC$ and $\triangle AEC=\triangle AoC$. With A as a center and a as a radius describe the arc EoF. With B as a center and b as a radius describe arc FOD. With C as a center and c as a radius, describe arc EoD. Join EF, FD and DE. Now $\angle EAF=2\angle CAB=120^\circ$. Therefore, EF is equal to a side of an equilateral triangle inscribed in a circle whose radius is a.

$\therefore$ $EF=a\sqrt{3}$. In same manner find $FD=b\sqrt{3}$ and $ED=c\sqrt{3}$. Area $\triangle EAF=196\sqrt{3}$, of $\triangle FBD=100\sqrt{3}$, of $\triangle DCE=240\frac{1}{4}\sqrt{3}$, of $\triangle EFD=823.18129+$. The area of the entire polygon $AFBDCE=1751.99357+$sq. rds. This is double the area of $\triangle ABC$.

$\therefore$ area $\triangle ABC=875.996785$ sq. rds. Equating the two expressions for the area, we have $\frac{1}{4}S^2\sqrt{3}=875.996785$. $\therefore$ $S=44.97+$ rods, side of the triangle.

23. **Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics, Indiana University, Bloomington, Indiana.**

Sum to n terms the following series: $11+25+45+71+103+\ldots.$

I. **Solution by B. F. BURLESON, Oneida Castle, New York, D. G. DORRANCE, Jr., Camden, New York, J. H. GROVE, Professor of Mathematics, in Howard College, Brownwood, Texas, and J A. TIMMONS, A. M., Professor of Mathematics in St. Mary's College, St. Mary's, Kentucky.**

The first differences are 14, 20, 26, 32, ...;
the second differences are 6, 6, 6, ...;
the third differences are 0, 0, ...

Putting d_1, d_2, d_3 for the first terms of the differences we have

$$d_1=14,\ d_2=6,\ d_3=0.$$

Then, by the "Differential Method," we have, if we put a for the first term of the series, and S for its sum,

$$S=na+\tfrac{1}{2}n(n-1)d_1+\tfrac{1}{6}n(n-1)(n-2)d_2,$$
$$=11n+7n(n-1)+n(n-1)(n-2),$$
$$=n^3+4n^2+6n=n(n^2+4n+6)=n[(n+2)^2+2].$$

Similarly solved by *P. S. BERG, J. A. CALDERHEAD, H. W. DRAUGHON, J. H. DRUMMOND, J. K. ELLWOOD, A. L. FOOTE, M. A. GRUBER, ARTEMAS MARTIN, F. P. MATZ, COOPER D. SCHMITT, H. C. WHITAKER*, and *G. B. M. ZERR.*

II. **Solution by J. F. W. SCHEFFER, Hagerstown, Maryland.**

This is an arithmetical progression of the 2d order, in which $a=11$,

$\triangle a=14$, $\triangle^2 a=6$, $\triangle^3 a=0$. $\therefore$ $S=11n+\binom{n}{2}14+\binom{n}{3}6=n(n^2+4n+6)$, or $n[(n+2)^2+2]$.

III. Solution by A. C. ROBERTS, Long Bottom, Ohio; C. E. WHITE, Trafalgar, Indiana; and OTTO GECKELER, Bloomington, Indiana.

Here $S=n[(n+2)^2+2]$, and this formula is derived as follows:

The series $11+25+45+71+103+141+185+\ldots\ldots$ consists of the sum of two series as follows:

1st $\quad 11[1+2+3+4+\ldots\ldots+n]$ whose sum$=11\left(\frac{n(n+1)}{2}\right)$

2d $\quad 3[1+4+3^2+16+5^2+\ldots\ldots+(n-1)^2]$ whose sum$=\frac{1}{2}[n(n-1)(2n-1)]$.

$\therefore$ the entire sum$=\frac{11}{2}[n(n+1)]+\frac{1}{2}[n(n-1)(2n-1)]=n^3+4n^2+6n$.

$$n^3+4n^2+6n=n[n^2+4n+6]=n[(n+2)^2+2].$$

IV. Solution by OTTO CLAYTON, A. B., Indiana University, Bloomington, Indiana, and JOHN B. FAUGHT, Bloomington, Indiana.

Making a table as follows:

$$11+\left\{\begin{matrix}11\\14\end{matrix}\right\}+\left\{\begin{matrix}11\\14\\14+6\end{matrix}\right\}+\left\{\begin{matrix}11\\14\\14+6\\14+6+6\end{matrix}\right\}+\left\{\begin{matrix}11\\14\\14+6\\14+6+6\\14+6+6+6\end{matrix}\right\},$$

we see that the nth or general term is $11+(n-1).14+\dfrac{(n-1)(n-2)}{2}.6$

$$S_n=11n+\Sigma_1^{n-1}(n-1).14+\Sigma_1^{n-1}\frac{(n-1)(n-2)}{2}.6=11n+\frac{n(n-1)}{2}.14+\left\{n-2\right.$$

$$\left.+\frac{(n-2)(n-3)}{2}.2+\frac{(n-2)(n-3)(n-4)}{6}\right\}.6=n(n^2+4n+6), \text{ or, } n[(n+2)^2+2].$$

V. Solution by W. WIGGINS, Richmond, Indiana, and the PROPOSER.

By inspection, the general term $U^n=3n^2+5n+3$.

Now take another series whose general term $V_n=n^3+(n-1)^2+3n$.

$$\triangle V_n=[(n+1)^3+n^2+3(n+1)]-[n^3+(n-1)^2+3n]=3n^2+5n+3=U_n.$$

By Chrystal, Chapter XXXI, Sec. 3, $\Sigma_{n=1}^{n}U_{n+1}=V_n-V_s\Sigma_{n=1}^{n}=U_n=V_n+1+V_1$ $=(n+1)^3+n^2+3(n+1)-4=n^3+4n^2+6n=n[n+2)^2+2]$.

PROBLEMS.

34. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics, Indiana University, Bloomington, Indiana.

$$\Sigma_1^n\frac{(n+2)^2}{n(n+4)}=\text{what?}$$

35. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Prove that the product of two numbers, each the sum of (4) squares may be expressed as the sum of four squares in 48 different ways and unite some or all of the 48 ways.

Solutions to these problems should be received on or before December 1st.

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

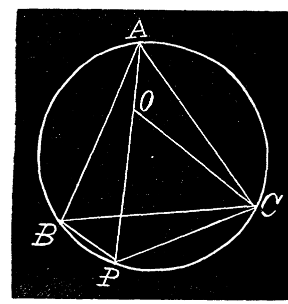
SOLUTIONS TO PROBLEMS.

19. Proposed by J. A. CALDERHEAD, Superintendent of Schools, Limaville, Ohio.

If any point be taken in the circumference of a circle, and lines be drawn from it to the three angles of an inscribed equilateral triangle, prove that the middle line so drawn is equal to the sum of the other two.

I. Solution by CHARLES E. MYERS, Canton, Ohio; P. S. BERG, Apple Creek, Ohio; Professor F. E. MILLER, Westerville, Ohio; R. H. YOUNG, Sunbury, Pennsylvania.

Let ABC be an equilateral triangle inscribed in a circle, P any point in the circumference; and PB, PC and PA lines drawn from the point to the three angles.



On PA take $PO=PC$, and join OC. The angles OPC and OPB, being measured by equal arcs, are equal and constant $=60^\circ$, and since $PO=PC$, the the triangle POC is equilateral and $OC=PC$.

In the triangles OCA and BCP we have $OC=PC$; $AC=BC$ and the angle ACO=angle BCP; therefore $AO=BP$. But $OP=PC$; therefore $BP+PC=PO+OA=PA$.

II. Solution by J. W. WATSON, Middle Creek, Ohio.

Let ABC be the inscribed equilateral triangle. P any point in the circumference. Join AP, CP and BP. We are to prove $AP=BP+CP$.

Put $a=$ a side of the triangle, $n=AP$, $z=BP$ and $s=CP$.

In the triangle APB, $a^2=n^2+z^2-nz$ since the angle APB is 60°.

In the triangle CPB, $a^2=s^2+z^2+sz$ since the angle CPB is 120°.

$\therefore n^2+z^2-nz=s^2+z^2+sz$, $n^2-s^2=sz+nz$ or $(n-s)(n+s)=z(n+s)$, $n-s=z$. $\therefore n=s+z$, or $AP=CP+BP$.

III. Solution by Professor J. R. BALDWIN, A. M., Davenport, Iowa, and T. A. SIMMONS, St. Mary's Kentucky.

Let P be any point in the circumference, ABC the inscribed equilateral triangle of which A and C lie next to P.

Draw AP and PC.

By a well known proposition $AB\times PC+AP\times BC=AC\times PB$, (1).

But $AB=BC=AC$, hence casting out the equal factor out of (1) we have $PC+AP=PB$. *Q. E. D.*

Also solved by *G. B. M. ZERR*, *J. H. DRUMMOND*, *M. A. GRUBER*, *J. K. ELLWOOD*, and *P. H. PHILBRICK*. ☞ Three other solutions *without names of contributors* were received.

21. Proposed by CHARLES E. MYERS, Canton, Ohio.

A cistern 6 feet in diameter contains 3 inches of water. If a cylinder, four

feet long and one foot in diameter, be laid in a horizontal position on the bottom, to what height will the water rise?

Solution by Professor G. B. M. ZERR, Principal of High School, Staunton, Virginia.

Let 2θ be the angle at the centre subtended by the chord made by the surface of the water.

Now since radius=6 inches and length=48 inches, area segment=36 $(\theta-\sin\theta\cos\theta)$; volume$=48\times36(\theta-\sin\theta\cos\theta)$; height of water$=6(1-\cos\theta)$. Volume also equals amount of rise in water. Volume$=\{6(1-\cos\theta)-3\}\pi(36)^2$.

$\therefore\ 48\times36(\theta-\sin\theta\cos\theta)=\{6(1-\cos\theta-3\}\pi(36)^2,\quad \therefore\ 4\theta-4\sin\theta\cos\theta=9\pi-18\pi\cos\theta.$

Now by double position, $\theta=63^\circ\ 13'\ 55.4''$.

$\therefore\ 6(1-\cos\theta)=3.29772.$

$\therefore$ water rises $3.29772-3=.29772$ inches.

Also solved by *SETH PRATT*, and *H. C. WHITAKER*.

22. **Proposed by J. A. TIMMONS, St. Mary's, Kentucky.**

Given, the perimeter of a triangle$=100(2s)$, the radius of the inscribed circle $=9(r)$, and the radius of the circumscribed circle$=(R)$; it is required to find (1) the sides of the triangle, (2) the radius of the circle circumscribing the triangle formed by bisecting the exterior angles of the original triangle, (3) the area of the triangle thus formed: all in terms of R, r, s.

I **Solution by H. C. WHITAKER, M. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

From the properties of the triangle,

$$a+b+c=2s$$
$$ab+ac+bc=s^2+r^2+4Rr$$
$$abc=4Rrs,$$

whence from the theory of equations, these quantities are the co-efficients of a cubic equation, the roots of which are the three sides. The numerical values are 32, 28.43225 and 39.56775.

Second. The length of the side (passing through A) of the outer triangle is $\dfrac{b\cos\frac{1}{2}C}{\cos\frac{1}{2}B}+\dfrac{c\cos\frac{1}{2}B}{\cos\frac{1}{2}C}$, its opposite angle being $\dfrac{B}{2}+\dfrac{C}{2}$. Dividing the side by twice the sine of this angle gives $\dfrac{b\cos^2\frac{1}{2}C+c\cos^2\frac{1}{2}B}{\sin B\cos^2\frac{1}{2}C+\sin C\cos^2\frac{1}{2}B}$ which by making $c=\dfrac{b\sin C}{\sin B}$ reduces to $b\div\sin B=2R$;—that is, the radius of the circle circumscribed about the outer triangle is twice the radius of the circle circumscribed about the inner triangle. The numerical value is 40.

Third. The length of the side passing through A of the outer triangle is also equal to $\dfrac{S}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$ with similar expressions for the other sides. Now the area of any triangle equals the product of the three sides divided by four times the radius of the circumscribed circle; that is

$$\text{Area}=\frac{S^2}{8R\cos^2\frac{1}{2}A\cos^2\frac{1}{2}B\cos^2\frac{1}{2}C}=2Rs$$

since $\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=\frac{s}{4R}$. The numerical answer is 2000.

II. **Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.**

Let ABC represent the triangle, and A', B', C' the centers of escribed circles. Applying opposite resp. to A, B, C; also I centre of inscribed circle. Denote the sides of the triangle $A\ B\ C$ by a, b, c, and its area by $\triangle$; then $a+b+c=2s$ (1), $sr=\triangle$, $\frac{abc}{4\triangle}=\frac{abc}{4sr}=R$, whence $abc=4Rrs$ (2). Squaring the equation $sr=\sqrt{s(s-a)(s-b)(s-c)}$, and putting for $a+b+c$, and abc their respective values, we obtain $ab+ac+bc=r^2+s^2+4Rr$ (3). Consequently the three roots of the cubic

$x^3-2sx^2+(r^2+s^2+4Rr)x-4Rrs=0$ (4) will furnish the three sides of the triangle. For the numerical values we obtain $32, 34\pm\sqrt{31}$.

Denoting the radii of the escribed circles A', B', C' resp. by S, S', S'', we have $A'B'=\frac{S+S'}{\cos\frac{1}{2}C}=\frac{\frac{\triangle}{s-a}+\frac{\triangle}{s-b}}{\cos\frac{1}{2}C}=\frac{\triangle}{\cos\frac{1}{2}C}\,\frac{c}{(s-a)(s-b)}=\frac{c\sqrt{ab}}{\sqrt{(s-a)(s-b)}}$

$=\frac{c}{\sin\frac{1}{2}C}=4R\cos\frac{1}{2}C$ (5) and by analogy $A'C'=4R\cos\frac{1}{2}B$ (6), $B'C'=4R\cos\frac{1}{2}A$ (7). In order to find the area of $\triangle A'B'C'$, we have $sA'B'C''=\frac{1}{2}A'B'.A'C'\sin A'=4R^2\cos\frac{1}{2}B\cos\frac{1}{2}C\sin A'=8R^2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=8R^2\frac{s}{4R}$ $=2Rs$ (8). Denoting the radius of the circle circumscribing $\triangle A'C'C''$ by R' we have $R'=\frac{A'B'\times A'C'\times B'C'}{8Rs}=\frac{64R^3\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{8Rs}$

$$=\frac{64R^3}{8Rs}\cdot\frac{s}{4R}=2R. \quad (9).$$

PROBLEMS.

32. **Proposed by Professor B. F. SINE, Shenandoah Normal College, Reliance, Virginia.**

If a given circle is cut by another circle passing through two fixed points the common chord passes through a fixed point.

33. **Proposed by T. JOHN COLE, Columbus, Ohio.**

A circular field contains 10 acres. A horse is tied to the fence with a rope sufficiently long to graze over one acre. Find length of the rope (1) when the horse is on the inside (2) when he is on the outside of the fence.

Solutions to these problems should be received on or before December 1st.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

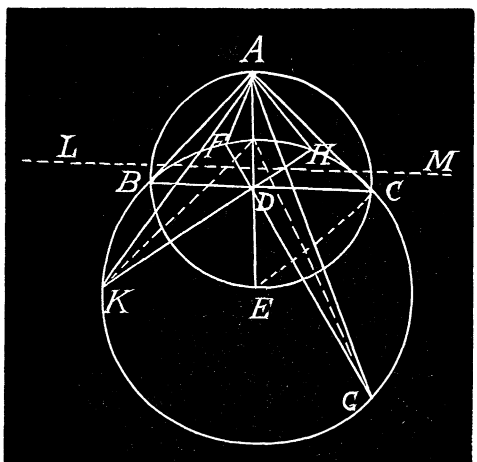
SOLUTIONS TO PROBLEMS.

3. Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania..

Given: the product of two sides of a triangle is 6000 the length of the bisector of the included angle is 60. Required: What is the maximum area, and what the greatest length of the third side.

II. A Geometrical construction by W. B. RIEGNER Philadelphia, Pennsylvania.

Make $AD=60$, $AE=\frac{6000}{60}=1000$, $EC=\sqrt{[DE^2+(AD\times DE)]}$. Any triangle with apex A and the third side a chord through D fulfills the conditions. No other triangle can. The centre of the circumscribing circle of



each of the triangles is on line LM. The bisectors of each triangle rest on arc BFC. The area swept over by the triangle, the bisector remaining fixed is 13921. Area$=1200\sqrt{6}$, 3rd side$=40\sqrt{10}$.

6. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Ohio.

A string is wound spirally twenty times around a cylinder 20 feet high and 2 feet in diameter. Through what distance will a dove fly in unwinding the string keeping it tense at all times (1) flying in the same plane and (2) not flying in the same plane?

II. Solution by G. B. M. ZERR, Principal of Schools, Staunton, Virginia.

Let $ABCD....$ be a prism of n sides inscribed in the cylinder $ABCDE$ of radius $OA=r$.

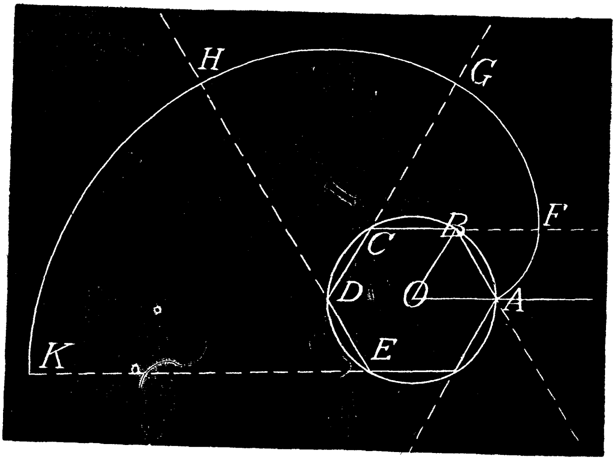
Then $\angle AOB=\angle ABF=\angle BCG=\&c=\frac{2\pi}{n}$ side $AB=2r\sin\frac{\pi}{n}$.

(1) Arc $AFGH\dots$ is a portion of the path of the dove, arc AF is described with radius $AB=2r\sin\frac{\pi}{n}$ and central angle $ABF=\frac{2\pi}{n}$. $\therefore$ length arc $AF=2r\times\frac{2\pi}{n}\sin\frac{\pi}{n}$. Similarly length of arc $FG=4r\times\frac{2\pi}{n}\sin\frac{\pi}{n}$, length of arc $GH=6r\times\frac{2\pi}{n}\sin\frac{\pi}{n}$.

The radius for nth sides of first revolution $=2nr\sin\frac{\pi}{n}$

The radius for nth side of second revolution $=4nr\sin\frac{\pi}{n}$

The radius for nth side of mth revolution $=2mnr\sin\frac{\pi}{n}$



$\therefore$ length of path $=S$ec $=2r\times\frac{2\pi}{n}\sin\frac{\pi}{n}\left\{1+2+3+\dots+(mn-1)\right\}$

$=2r\times\frac{2\pi}{n}\sin\frac{\pi}{n}\times\frac{mn(mn-1)}{2}=2\pi rm(mn-1)\sin\frac{\pi}{n}$.

Now when n becomes infinite the prism becomes a cylinder and $n\sin\frac{\pi}{n}=\pi r$ also $\sin\frac{\pi}{n}=0$.

$\therefore$ $S=2\pi^2r^2m^2$. In this case $r=1$, $m=20$.

$\therefore$ $S=2(20)^2\pi^2=800\pi^2=7895.72$ feet approximately.

(2) In this case let the string cross the edge of the prism at a distance d apart. Then since there are n sides, one side falls a distance $\frac{d}{n}$ and the length of the string on one face $=\sqrt{4r^2\sin^2\frac{\pi}{n}+\frac{d^2}{n^2}}$ the rest of the notation is the same as in case one since the string is kept horizontal.

$\therefore$ length $=S=\frac{2\pi}{n}\sqrt{4r^2\sin^2\frac{\pi}{n}+\frac{d^2}{n^2}}\left\{1+1+3\dots+(mn-1)\right\}$

$$=\frac{2\pi}{n}\sqrt{4r^2\sin^2\frac{\pi}{n}+\frac{d^2}{n^2}}\,\frac{mn(mn-1)}{2}=\pi m(mn-1)\sqrt{4r^2\sin^2\frac{\pi}{n}+\frac{d^2}{n^2}}$$

$=\pi m^2\sqrt{4\pi^2r^4+d^2}$, when n is infinite but $r=1$ and $m=20$, $d=1$.

$\therefore S=400\pi\sqrt{(4\pi^2+1)}=7995.12$ feet approximately.

[The above revised solution of Professor Zerr's now agrees in result with that of Professor Hume previously published.—Editor.]

17. Proposed by H. W. DRAUGHON, Clinton, Louisiana

Find the volume generated by revolving a circular segment whose base is a given chord ($2a$), about any diameter as an axis.

II. First Solution by F. P MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Let $\angle ECH=A$, $CA=r$, $AE=a$, $CE=x$, and $\angle ACD=\theta$; then $x'=r\cos\theta=\sqrt{(r^2-a^2)}$, and the required volume generated becomes

$$V=2(2\pi x\sin A)\int_{x'}^{r}\sqrt{(r^2-x^2)}dx=4\pi\sin A\left[-\tfrac{1}{3}\left(r^2-x^2\right)^{\frac{3}{2}}\right]_{x'}^{r}=\tfrac{4}{3}\pi a^3\sin A.$$

SECOND SOLUTION.

Put $\angle CAE=\phi$; then $x=r\sin\phi$, $dx=r\cos\phi d\phi$, and $\sqrt{(r^2-x^2)}=r\cos\phi$. The limits of integration are $\frac{1}{2}\pi$ and $\cos^{-1}(a/r)=\phi'$; and after obvious transformations, we have

$$V=4\pi\sin A\int_{\phi'}^{\frac{1}{2}\pi}r^3\cos^2\phi\sin\phi d\phi=4\pi\sin A\left[-\tfrac{1}{3}r^3\cos^3\phi\right]_{\phi'}^{\frac{1}{2}\pi}=\tfrac{4}{3}\pi a^3\sin A.$$

THIRD SOLUTION.

Represent the center of gravity of the segment $ADBE$ by G; then from Mechanics, we have $CG=\dfrac{\frac{1}{12}(2a)^3}{r^2\sin^{-1}(a/r)-a\sqrt{(r^2-a^2)}}$, in which the denominator is the area of the segment $ADBE$, $=\mathbf{A}$. Since $FG=CG\sin A=(2a^3/3\mathbf{A})\sin A$, the *Centrobaryc Method* gives for the volume generated $V=2\pi(2a^3/3\mathbf{A})\sin A\times\mathbf{A}=\frac{4}{3}\pi a^3\sin A$.

[In the solution by the Proposer, previously published the lower integral should have been $\sqrt{(R^2-a^2)}$. There was a misprint.—Editor.]

Note on Prob. No. 13 by JOHN DOLMAN, Jr, Philadelphia, Pennsylvania.

"It may be worth noting that the solution of No. 13 is incomplete in failing to state that the value of x found makes y a minimum only when angle EAD equals or is less than $\cos^{-1}\frac{4}{5}$ [considering the angle negative when D is west of E]. In all other cases the value, $x=\sin^{-1}\frac{4}{5}$, makes y a maximum and the steamers could *meet* in two possible points. All this, together with the critical value found, may be seen from inspection without any analytical work."

PROBLEMS.

27. Proposed by G. B. M ZERR. A. M. Principal of Schools, Staunton, Virginia.

A runs around the circumference of a circular field with velocity m feet; B starts from the centre with velocity $n>m$ feet to catch A. The straight line joining their positions always passes through the centre. Find the equation to the curve described by B, the distance he runs and the time occupied.

MECHANICS.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

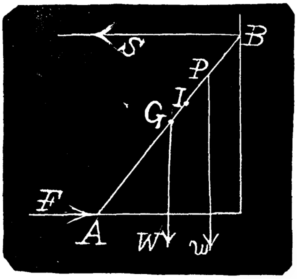
SOLUTIONS TO PROBLEMS.

9. Proposed by CHARLES E. MYERS, Canton, Ohio.

A ladder inclined at an angle of 60° to the horizon rests with one end on a *rough* pavement and the other against a *smooth* vertical wall. If the coefficient of friction between the foot of the ladder and the pavement is $\frac{1}{6}\sqrt{3}$, to what height can a man ascend before the ladder will begin to slip?

Solution by P. F. FINKEL, Professor of Mathematics, Physics, and Astronomy, Kidder Institute, Kidder, Missouri.

Let AB be the ladder, length $2l$; θ its inclination to the horizon, W its weight and G its center of gravity; $AP=x$, the distance the man ascends and w his weight; I, the center of gravity of the man and ladder; $F=\mu$, the coefficient of friction; and S and R the normal resistance at A and B, respectively.



Then we have $S-F=0\ldots.(1)$ and, $W+w-R=0\ldots.(2)$. Taking moments about I, we have $W.GI=w.PI$, or $W.GI-w.\ PI=0\ldots.(3)$. But $GI+PI=x-\frac{1}{2}l$. Hence, from (3), $GI=\frac{w}{W+w}(x-\frac{1}{2}l)$. Taking moments about A, $(W+w).AI=AB.\ R(\cos\theta-\mu\sin\theta)$ or $(W+w)\left[\frac{w}{W+w}(x-\frac{1}{2}l)+\frac{1}{2}l\right]\cos\theta=2lR(\cos\theta-\mu\sin\theta)\ldots.(4)$. (2) and (4) give $2l(W+w)(\cos\theta-\mu\sin\theta)=(Wl+wx)\cos\theta$, whence

$$x=\frac{2l(W+w)(1-\mu\tan\theta)-Wl}{w}.$$ But $\mu=\frac{1}{6}\sqrt{3}$ and $\tan\theta=\sqrt{3}$.

$\therefore\ AP=x=l$. Hence the man can ascend to the middle point of the ladder.

[Note.—This problem and its solution was selected for publication in the Aug. No. But it was lost in the office of the publishers and thus we are unable to credit our contributors for their solutions to it. ED.]

10. Proposed by G. B. M ZERR, A. M., Principal of High School, Staunton Virginia.

A paraboloid floats in a liquid which fills a fixed paraboloid shell; both the paraboloid and the shell have their axis vertical and their vertices downward; the latus rectum of the paraboloid and shell are equal,and the axis of the shell is m times that of the paraboloid. If the paraboloid be pressed down until its vertex reaches the vertex of the shell. so that some of the liquid overflows, and then released, it is found that the paraboloid rises until it is just wholly out of the liquid, and then begins to fall. Prove that (1) the densities of the paraboloid and liquid are in the ratio.

$2[m^2+m+1=(m+1(\sqrt{m^2}-1)]\ :\ 3\sqrt{(m+1)\div(m-1)}$, the free surface of the liquid being supposed to remain horizontal throughout the motion; and (2)

if cone and conical be used, is the ratio $3\ [m^4-1-(m^3-1)\sqrt[3]{(m^3-1)}: 4\sqrt[3]{(m^3-1)}$, the vertical angles being equal.

Solution by G. B. M. ZERR, Principal of High School, Staunton, Virginia.

In the two positions in which the velocity of the paraboloid is zero, the heights of the centre of gravity of the paraboloid and liquid are equal.

Let ρ be the density of the paraboloid, σ of the liquid, h the height of fhe paraboloid.

$y^2=4ax$ the equation to the parabola of revolution which generates the paraboloid; x the height of the surface of the liquid in the second position.

Then $2a\pi x^2=2a\pi m^2h^2-2a\pi h^2=2a\pi(m^2-1)h^2$.

$\therefore\ x=\sqrt{m^2-1}\ h.$

Hence $\frac{2}{3}.mh.2a\pi m^2h^2\sigma-\frac{2}{3}h.2a\pi h^2(\sigma-\rho)=\frac{2}{3}\sqrt{m^2-1}\ h.2a\pi(m^2-1)h^2\sigma$ $+\{\sqrt{m^2-1}+\frac{2}{3}\}h.2a\pi h^2\rho.$

$\therefore\ \frac{2}{3}m^3\sigma-\frac{2}{3}(\sigma-\rho)=\frac{2}{3}\sqrt{(m^2-1)^3}\sigma+(\sqrt{m^2-1}+\frac{2}{3})\rho.$

$$2\{m^3-1-\sqrt{(m^2-1)^3}\}\sigma=3\sqrt{m^2-1}\ \rho,$$

$$2\{m^2+m+1-(m+1)\sqrt{m^2-1}\}\sigma=3\sqrt{(m+1)\div(m-1)}\rho.$$

$\therefore\ \rho:\sigma=2\{m^2+m+1-(m+1)\sqrt{m^2-1}\}:3\sqrt{(m+1)\div(m-1)}.$

For the cone and conical cup, we have $\frac{1}{3}\pi x^2\tan^2\beta=\dfrac{\pi m^3h^3}{3}\tan^2\beta$ $-\dfrac{\pi h^3}{3}\tan^2\beta$ where β is the semi-vertical angle of the cone.

$\therefore\ x=h\sqrt[3]{m^3-1}.$

Hence $\frac{3}{4}.mh.\dfrac{\pi m^3h^3}{3}\tan^2\beta\sigma-\frac{3}{4}h.\dfrac{\pi h^3}{3}\tan^2\beta\,(\sigma-\rho)$

$=\frac{3}{4}h\sqrt[3]{(m^3-1)}\ \dfrac{\pi(m^3-1)h^3}{3}\tan^2\beta\sigma+\{\sqrt[3]{(m^3-1)}+\frac{3}{4}\}h.\dfrac{\pi h^3}{3}\tan^2\beta\rho.$

$\therefore\ \frac{3}{4}m^4\sigma-\frac{3}{4}(\sigma-\rho)=\frac{3}{4}(m^3-1)\sqrt[3]{m^3-1}\ \sigma+\{\sqrt[3]{m^3-1}+\frac{3}{4}\}\rho.$

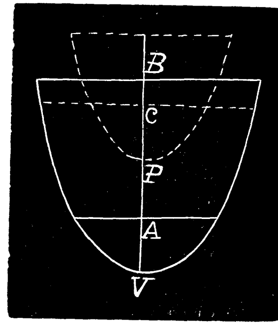
$\therefore\ 3\{m^4-1-(m^3-1)\sqrt[3]{m^3-1}\}=4\sqrt[3]{m^3-1}\ \rho.$

$\therefore\ \rho:\sigma=3\{m^4-1-(m^3-1)\sqrt[3]{m^3-1}\}:4\sqrt[3]{m^3-1}.$

II. Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics in the University of Mississippi.

Let $VA(=a)$ be the axis of the paraboloid; $VB(=ma)$, that of the shell; $2p$, their common latus rectum; ρ, the ratio of the density of the solid to

to that of the liquid; P, the vertex of the paraboloid at time t, the depth of the liquid being Vc; $VP=x$, $Pc=y$.



The paraboloid rises with constant acceleration $\frac{1-\rho}{\rho}g$ and, therefore, its upper surface reaches that of the liquid with velocity $\sqrt{2ga(m-1)\frac{1-\rho}{\rho}}$. The acceleration now becomes variable. As the solid continues to rise the level of the liquid is lowered. The upward pressure on the paraboloid is always proportional to y^2. y can be found from the equation

$$\pi p[m^2a^2-(x+y)^2]=\pi p(a^2-y^2).$$

$$y=\frac{m^2a^2-a^2-x^2}{2x}.$$

The equation of motion is $\frac{d^2x}{dt^2}=cy^2-g$, where c is a constant.

Substituting the value of y, multiplying by $2dx$, and integrating,

$$\left(\frac{dx}{dt}\right)^2=\frac{c}{2}\left[\frac{x^3}{3}+2a^2(1-m^2)x+\frac{a^4(2m^2-m^4-1)}{x}\right]-2gx+c_1.$$

c can be found from the first differential equation since, when $y=a$,

$$\frac{d^2x}{dt^2}=\frac{1-\rho}{\rho}g.$$

Hence $c=\frac{g}{a^2\rho}$. c_1 can be determined from the second differential equation for, when $x=a(m-1)$, $\left(\frac{dx}{dt}\right)^2=2ga(m-1)\frac{1-\rho}{\rho}$.

$$c_1=\frac{4ga}{3\rho}(m-1)(m^2+m+1).$$

Substituting these values of c and c_1,

$$\left(\frac{dx}{dt}\right)^2=\frac{g}{2a^2\rho}\left[\frac{x^3}{3}-2a^2(m^2-1)x-\frac{a^4(m^2-1)^2}{x}\right]-2gx+\frac{4ga}{3\rho}(m-1)$$

(m^2+m+1). $\frac{dx}{dt}=0$ at the instant that the paraboloid is wholly without the liquid. At this time $y=0$, and, therefore, $x=a\sqrt{m^2-1}$.

Substituting these values and solving for ρ,

$$\rho=\frac{2[(m-1)(m^2+m+1)-(m^2-1)^{\frac{3}{2}}]}{3(m^2-1)^{\frac{1}{2}}}=\frac{2[m^2+m+1-(m+1)\sqrt{m^2-1}]}{3\sqrt{\frac{m+1}{m-1}}}$$

2nd. The velocity with which the base of the cone reaches the free surface of the liquid is $\sqrt{2ga(m-1)\frac{1-\rho}{\rho}}$, as before. Afterwards the bouyant

force is proportional to y^3 and y may be found from $\frac{\pi \tan^2 \infty}{3}[m^3a^3-(x+y)^3]$

$=\frac{\pi \tan^2 \infty}{3}(a^3-y^3)$ where ∞ is the semi-vertical angle of the cone.

$$y=-\frac{x}{2}+\sqrt{\frac{a^3(m^3-1)}{3x}-\frac{x^2}{12}}.$$

The equation of motion is $\frac{d^2x}{dt^2}=cy^3-g.$

Since, when $y=a$, $\frac{d^2x}{dt^2}=\frac{1-\rho}{\rho}g$, $c=\frac{g}{a^3\rho}$.

Substituting this and the value of y

$$\frac{d^2x}{dt^2}=\frac{g}{a^3\rho}\left[\left(\frac{a^3(m^3-1)}{3x}+\frac{2x^2}{3}\right)\sqrt{\frac{a^3(m^3-1}{3x}-\frac{x^2}{12}}-\frac{a^3(m^3-1)}{2}\right]-g.$$

Integrating,

$$\left(\frac{dx}{dt}\right)^3=\frac{2g}{a^3\rho}\left[-2x\left(\frac{a^3(m^3-1)}{3x}-\frac{x^2}{12}\right)^{\frac{3}{2}}-\frac{a^3(m^3-1)}{2}x\right]-2gx+c_1.$$

When $x=a(m-1)$, $\left(\frac{dx}{dt}\right)^2=2ga(m-1)\frac{1-\rho}{\rho}$.

$\therefore\ c_1=\frac{3ag}{2\rho}(m^4-1).$

When $y=0$ and, consequently, $x=a\sqrt[3]{m^3-1}$, $\frac{dx}{dt}=0$.

$$\therefore\ 0=\frac{2g}{a^3\rho}\left(-\frac{3a^4(m^3-1)^{\frac{4}{3}}}{4}\right)-2ga\sqrt[3]{m^3-1}+\frac{3ag}{2\rho}(m^4-1), \text{ and}$$

$$\rho=\frac{3[m^4-1-(m^3-1)\sqrt[3]{m^3-1}]}{4\sqrt[3]{m^3-1}}.$$

PROBLEMS.

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16. Proposed by A. H. BELL, Hillsboro, Illinois.

An iron bar 20 feet long and weighing 2,000 lbs. leans against a wall at angles 30°, 45°, and 80°. How much pressure does the floor and wall receive?

17. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the law of density of strings collected into a heap at the edge of a table with the end just over the edge, so that equal masses may always pass over in equal times.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

A Note to Solutions of Problem No. I. by J. H. DRUMMOND, LL. D., Portland, Maine.

I assumed, as I perceive erroneously, that solutions in *integral* numbers were required; hence my remarks that "there are *comparatively* few square numbers" which can be divided into two squares.

Mr. Adcock's solution is a *very fine* one, but it seems to me that it can be put into simpler and much more easily remembered forms, thus $(p^2+q^2)^2=(p^2-q^2)^2+(2pq)^2$. Hence $1=\left(\frac{p^2-q^2}{p^2+q^2}\right)^2+\left(\frac{2pq}{p^2+q^2}\right)^2$ and $n^2=n^2\left(\frac{p^2-q^2}{p^2+q^2}\right)^2+n^2\left(\frac{2pq}{p^2+q^2}\right)^2$, in which p and q may be any unequal numbers; to obtain prime numbers, however, p and q must be prime to each other and one odd and the other even.

2. Proposed by J. M. COLAW, Principal of High School, Monterey, Virginia.

Find two numbers, such that the difference of their squares may be a cube, and the difference of their cubes a square.

III. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, P. C.

Let ax^3+b and ax^3-b denote the numbers. The difference of their squares is $4abx^3$, which must be a cube, $=8a^3x$ say, then $b=2a^2$. The difference of their cubes is $6a^2bx^6+2b^3$, which must be a square; or, substituting $2a^2$ for b and then striking out the square factor $4a^4$, ^{3} $3x^6+4a^2=\square=(2x^3-a)^2$ suppose; whence $x^3=8a$. Take $a=1$, then $x=2$, $b=2$; and the numbers are 10 and 6.

10. Proposed by L. B. HAYWARD, Bingham, Ohio.

Find two numbers such that each of them, and also their sum and their difference, when increased by unity shall all be squares.

III. Solution by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let x^2-1 and y^2-1 denote the numbers required and the first two conditions are satisfied in the notation.

But we still have $x^2+y^2-1=\square=v^2$, and $x^2-y^2+1=\square=w^2$, to dispose of. From the first of these we get $y^2=v^2-x^2+1$, and by adding the first and second we get $2x^2=v^2+w^2$.

Let $u=t+u$, $w=t-u$ and the last equation becomes $x^2=t^2+u^2$, which is satisfied by $t=n(p^2-q^2)$, $u=2npq$, and then $x=n(p^2+q^2)$, $v=n(p^2+2pq-q^2)$.

Substituting these values of x and v in $y^2=v^2-x^2+1$ we get

$y^2=4npq(p^2-q^2)+1=\square=(2mn-1)^2$ say; whence $n=\frac{m}{m^2-pq(p^2-q^2)}$, where m, p, q may be chosen at pleasure.

If $p=3$, $q=2$, then $n=\frac{m}{m^2-30}$, $=1$ when $m=6$; $\therefore$ $x=13$, $y=11$, and the numbers are 168 and 120.

If $p=4$, $q=1$, then $n=\frac{m}{m^2-60}$, $=2$ whem $m=8$; and then $x=34$, $y=31$, and the numbers are 1155 and 960.

PROBLEMS.

16. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

Find three numbers such that the cube of any one plus the sum of the squares of the other two will be a square.

17. Proposed by ARTEMAS MARTIN, LL. D., U. S Coast and Geodetic Survey Office, Washington, D. C.

Is it possible to find two positive whole numbers such that each of them, and also their sum and their difference, when *diminished* by unity shall all be squares?

Solutions to these problems should be received on or before December 1st.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

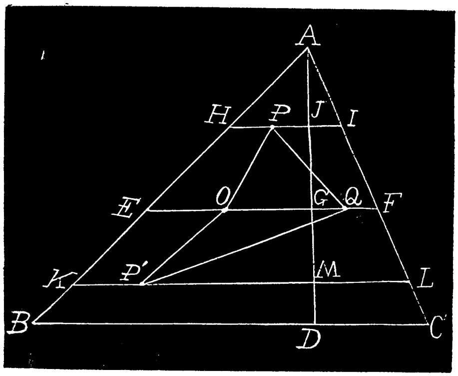
SOLUTIONS TO PROBLEMS.

8. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Prove that the mean area of all triangles having their vertices upon the surface of a given triangle and bases parallel to the base of the given triangle, is $\frac{13}{210}$ (area of given triangle).

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Represent AD by a, BC by b, and the area of $\triangle ABC$, $=\frac{1}{2}ab$, by $\triangle$.



Draw the random line EF, and on it choose at random the two points O and Q. Take any point P in the $\triangle\ AEF$, and complete the $\triangle\, OPQ$ the mean area of which is to be found. The point P may also be taken in the trapezoid $BCFE$, and then represented by P'. Put $AJ=y$, $AG=x$, $HP=z$, $EO=v$, $EQ=w$, $HI=n=(b\div a)\,y$, and $EF=m=(b\div a)x$; then will $GD=a-x$, $JG=x-y$, $GM=y-x$, $OQ=w-v$, $\triangle\, OPQ=\frac{1}{2}(x-y)(w-v)$, and $\Delta\, OP'Q=\frac{1}{2}(y-x)(w-v)$.

The required mean area, therefore, becomes

$$\Delta=\frac{\frac{1}{2}\int_0^a\int_0^m\int_0^w\left[\int_0^x\int_0^n(x-y)dydz+\int_x^a\int_0^n(y-x)dydz\right](w-v)dxdwdv}{\int_0^a\int_0^m\int_0^w\left[\int_0^x\int_0^n dydz+\int_x^a\int_0^n dydz\right]dxdwdv}$$

$$=\tfrac{1}{2}\left(\frac{12}{a^2b^3}\right)\left(\frac{b}{6a}\right)\int_0^a\int_0^m\int_0^w[2(a^3+x^3)-3a^2x](w-v)dxdwdv$$

$$=\frac{1}{2a^3b^2}\int_0^a\int_0^m[2(a^3+x^3)-3a^2x]dxw^2dw$$

$$=\frac{b}{6a^6}\int_0^x[2(a^3+x^3)-3a^2x]dx=\frac{13}{210}\left(\frac{ab}{2}\right)=\frac{13}{210}\triangle,$$ which is the result given in the problem.

II. Solution by the PROPOSER.

Let ABC be the given triangle, MNP the triangle whose average area is required, having its base NP parallel to AB the base of the given triangle. Draw GMG' and CL parallel to AB.

Let $CH=u$, $CG=v$, $HN=x$, $HP=y$, $GM=z$, $GG'=z'$, $HH'=x'$. Then we have area $MNP=\frac{1}{2}(x-y)(u-v)\sin A$, when $v<u$

area $MNP=\frac{1}{2}(x-y)(v-u)\sin A$, when $v>u$.

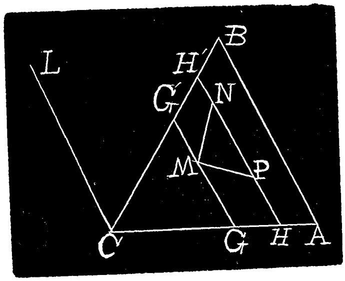
The limits of u are 0 and b; of v, 0 and u, and u and b; of x, 0 and $x'=\frac{cu}{b}$; of y, 0 and x; of z, 0 and $z'=\frac{cu}{b}$.

Hence the required average area is

$$\Delta=\frac{\int_0^b\int_0^{x'}\int_0^x\left\{\int_0^u\int_0^{z'}\frac{1}{2}(x-y)(u-v)\sin A\,dvdz+\int_0^b\int_0^{z'}\frac{1}{2}(x-y)(v-u)\sin A\,dvdz\right\}dudxdy}{\int_0^b\int_0^{x'}\int_0^x\int_0^b\int_0^{z'}dudxdydvdz}$$

$$=\frac{6\sin A}{b^2c^3}\int_o^b\int_o^{x'}\int_o^x\left\{\int_o^u\int_o^{z'}(x-y)(u-v)dvdz\right.$$

$$\left.+\int_o^b\int_o^{z'}(x-y)(v-u)dvdz\right\}dudxdy$$



$$=\frac{6\sin A}{b^3c^2}\int_o^b\int_o^{x'}\int_o^x\left\{\int_o^u(x-y)(uv-v^2)dv\right.$$

$$\left.+\int_o^b(x-y)(v^2-uv)dv\right\}dudxdy$$

$$=\frac{\sin A}{b^3c^2}\int_o^b\int_o^{x'}\int_o^x(2u^3+2b^3-3b^2u)(x-y)dudxdy$$

$$=\frac{\sin A}{2b^3c^2}\int_o^b\int_o^{x'}(2u^3+2b^3-3b^2u)x^2dudx$$

$$=\frac{c\sin A}{6b^6}\int_o^b(2u^6+2b^3u^3-3b^2u^4)du$$

$$=\frac{13bc\sin A}{420}=\frac{13}{210}\tfrac{1}{2}(bc\sin A)=\frac{13}{210}\text{(area of given triangle)}$$

9. **Proposed by H. C. WHITAKER, B. S., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.**

Four numbers taken at random are multiplied together. What is the probability that the last digit will be 0?

I. **Solution by H. W. DRAUGHON, Clinton, Louisiana.**

The probability that the final digit will be odd is $(\frac{5}{10})^4=\frac{625}{10000}$; the probability that it will be 2, 4, 6, or 8, is $(\frac{4}{10})^4+4(\frac{4}{10})(\frac{4}{10})^3+6(\frac{4}{10})^2(\frac{4}{10})^2+4(\frac{4}{10})^3(\frac{4}{10})=\frac{15(4)^4}{10000}=\frac{3840}{10000}$. $\therefore$ the probability that it will be 0 is, $P=1-\frac{3840}{10000}-\frac{625}{10000}=\frac{5535}{10000}=\frac{1107}{2000}$.

II. **Solution by F P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

We know from *Hall and Knight's Higher Algebra* if n integers be taken at random and multiplied together, the probability that the last digit of the product is 1, 3, 7, or 9, is $P_1=\frac{4^n}{10^n}$; also, the probability that the last digit of the product is 2, 4, 6, or 8, is $P_2=\frac{8^n-4^n}{10^n}$; and, finally, the probability that the last digit of this product is 5, is $P_3=\frac{5^n-4^n}{10^n}$. Consequently the probability that the last digit of this product is *zero*, when $n=4$, is $P_4=1-(P_1+P_2+P_3)$; that is, $P_4=\frac{10^4-8^4-5^4+4^4}{10^4}=\frac{(5^4-4^4)(2^4-1)}{5^4\times2^4}=\frac{1107}{2000}$.

Solutions to this problem were also received from *Hon. JOSIAH DRUMMOND, P. H. PHILBRICK* and *J. F. W. SCHEFFER.*

PROBLEMS.

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18. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.**

A surface one inch square is thrown at random upon a surface one foot square, but so as always to lie wholly upon the larger surface. Find the mean value of *the sum of the distances* of the vertices of the smaller surface, from *any* vertex of the larger surface.

19. **Proposed by H. W. DRAUGHON, Clinton, Louisiana.**

From one corner of a square field, a boy runs in a random direction, with a random uniform velocity. The greatest distance the boy can run in one minute is equal to the diagonal of the field. What is the probability that the boy will be in the field at the end of one minute?

Solutions to these problems should be received on or before December 1st.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

4. **Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.**

I have two grindstones, each $\frac{1}{2}$ inch thick. One is 6 in. and the other $4\frac{1}{2}$ in. in diameter, the aperature at center of each being $1\frac{1}{2}$ in. If when in motion they are continually tangent to each other, and $\frac{1}{4}$ cu. in. is ground off the larger wheel and $\frac{1}{4}$ cu. in. off the smaller in the first hour, how must their speed be increased so that the same amount per hour may be ground off each wheel until one is worn out? If in the first hour the larger wheel makes a revolutions, and the smaller b, how many must each make in each succeeding hour?

II. **Solution by P. H. PHILBRICK, C. E., Lake Charles, Louisiana.**

The diagonal of the aperature is, $\sqrt{4.5}=2.1213$. Vol. of larger stone outside of the circle circumscribing the aperature is, $\frac{1}{2}\frac{\pi}{4}(36-4.5)=12.37$ and the same for the smaller stone is, $\frac{1}{2}\frac{\pi}{4}(20.25-4.5)=6.185$. The stones will therefore wear out at the same time and in $12.37\div\frac{1}{2}=24.74$ hours.

The side face of larger stone including the aperature$=\frac{\pi}{4}36=28.27$; and the available area$=\frac{\pi}{4}(36-4.5)=24.74$. One inch area is therefore worn off the side face of the larger stone per hour and one half of an inch off the face of the smaller stone. Hence, including the aperature, the side face of the larger stone at the end of the first hour is 27.27.

At end of second hour 26.27.

At end of third hour 25.25.

etc., etc.

At the end of the $24\frac{1}{2}$ hour is 4.27.

The areas of the circumscribing squares at end of successive hours differ by $1\div\frac{\pi}{4}=\frac{4}{\pi}$. Hence letting D=the diameter of stone at the beginning; D_1, D_2, etc., the same at the end of the 1st, 2d hours, etc. we have,

$$D=\sqrt{36}\quad=6$$

$$D_1=\sqrt{36-\frac{4}{\pi}}=5.8929$$

$$D_2=\sqrt{36-\frac{8}{\pi}}=5.7839$$

$$D_3=\sqrt{36-\frac{12}{\pi}}=5.6728$$

etc., etc.

Finally, $D_{24}=\sqrt{36-\frac{96}{\pi}}=2.3329.$

Hence diameter worn off 1st hour$=6-5.8929=0.1071$,
and the 2d hour$=5.8929-5.7839=0.1090$,
and the 3rd hour$=5.7839-5.6728=0.1111$,
etc. etc.

The diameter worn off after the end of 24 hours is, $2.3329-2.1213=0.2116$.

Now $\frac{1}{2}\frac{\pi}{4}[(2.3329)^2-(2.1263)^2]=0.37$ of a cubic inch as it ought.

In 24 hours, $24\times\frac{1}{2}=12$ inches are ground off, leaving $12.37-12=0.37$ of an inch, as just found, and this will consume $0.37\div\frac{1}{2}=.74$ of an hour, or 24.74 hours in all.

In the same way the diameters worn off the smaller stone are found.

The number of revolutions per hour are precisely as the diameters ground off and are therefore shown above.

[Prof. Philbrick solves for square apertaure and in part under a slightly different interpretation from Prof. Hume, whose solution was previously published. We publish above for comparison. H. W. Draughon's solution agreed with Prof. Hume's, except that it was in general terms.—Editor.]

6, **Proposed by H. C. WHITAKER, B S., M. E., Professor of Mathematics, Manual Training School, Philadelphia. Pennsylvania.**

Two men wish to buy a grindstone 42 inches in diameter and one foot thick at the center. To what thickness at the outer edge should the stone uniformly taper from the center so that each man may grind off 18 inches of the diameter and both have equal shares, the central six inches of the diameter being waste?

III. Solution by the PROPOSER.

[Refer to fig. used on pg. 174 of the May number.]

Let equation of line generating the lateral surface of the stone be $x=2a-3by$.

Then $\text{Vol}=\int 4\pi xydy=4\pi\int(2ay-3by^2)dy=4\pi(ay^2-by^3)$

Taking first the limits to be 21 and 12 and then 12 and 3 and equating

we have $297a-7533b=135a-1701b$ or $a=36b$. But $a=3$ and therefore $b=\frac{1}{12}$.

Substituting these values and $y=21$ in the first equrtion $x=\frac{3}{4}$ making the width $1\frac{1}{2}$ inches.

PROBLEMS.

18. Proposed by M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Indiana.

Show generally that a system of confocal conics is self orthogonal. [From Johnson's Differential Equations.]

19. Propoped by F. P MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A spherical shrapnel-shell moving in a practically horizontal direction with a constant velocity $V_1=1500$ feet per second and at a uniform height $h=\frac{1}{2}g$ feet, explodes *equably* scattering the inclosed balls and fragments with a uniform velocity $V_2=1200$ feet per second. Draw the curve bounding the *minimum surface on the earth* on which the inclosed balls and fragments of the shell have fallen.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

Definitions of a Fraction. A Query by "H. C. K."

Three views of what a fraction is are here given, with the request that every reader send the editor of this department a card stating his preference.

(A). A fraction is one or more of the equal parts into which "a unit" or anything is divided; as, $\frac{m}{n}$. When $m>n$, the fraction is called "improper".

(B). A fraction is a number of the equal parts of one or more like units or things. The fraction $\frac{m}{n}$ is proper whether $m>n$ or $m<n$.

(C). A fraction is any number of the equal parts into which anything is divided, less than the whole number of such parts. When $m=n$, $\frac{m}{n}$ is a unit. When $m>n$, $\frac{m}{n}$ is a mixed number in fractional form,—not a fraction at all, since we can not take m equal parts of a thing divided into only n parts.

"An Unreasonable Rule" Again. By Professor P. H PHILBRICK.

Mr. Pratt fails to make out his case. He does show indeed that the rate of convergency in the vicinity of the parallel of 50° latitude is nearly one-fifth more than near the parallel of 40° latitude but no one, so far as I know,

has doubted that, for these parallels are nearly 700 miles apart, whereas across four townships is but 24 miles, or about one thirtieth as far.

For 20′, or say 23 miles, on each side of the parallel of 40°, we have as follows:

$$\begin{array}{ll} \cos 39^\circ\ 20'=.76977 & \cos 40^\circ\ 00'=.76604 \\ \cos 40^\circ\ 00'=\underline{.76604} & \cos 40^\circ\ 20'=\underline{.76229} \\ \text{Difference}=.00373 & .00375 \end{array}$$

These results show the convergency for 23 miles on one side of latitude 40°, is practically the same as on the other side. So generally for other points.

Curious Process for Maximum or Minimum, by EDMUND FISH, Hillsboro, Illinois.

Example 1. A farmer says to his son, we are now ready to measure off and fence in the ten acres for our new orchard. The east side is adjacent to the Railroad and is already fenced. On the north it joins neighbor Brown, and he agrees to make half that fence. So the south side, the west side and half the north side are all we have to make. Now I propose to make the length and breadth just such as can be inclosed with the least fence, and as soon as you have figured that out, we will set about it. John takes his pencil and remarks,—The width of the field shall be x rods. Then the length will be $\frac{1600}{x}$, and the fence required will be expressed by $\frac{3x}{2}+\frac{1600}{x}$.

For a second member, for want of anything better, continues John, we will place this expression equal to itself. The farmer smiles at this sage proposal, for he too knows something of algebra, but John goes on, $\frac{3x}{2}+\frac{1600}{x}=\frac{3y}{2}+\frac{1600}{y}$, in which x and y represent the same thing.

Reducing gives, $\quad 3x^2y+3200y=3xy^2+3200x$,

$$3(x^2y-xy^2)=3200\ (x-y),$$

$$3xy=3200, \text{ and } x \text{ and } y \text{ being identical,}$$

$3x^2=3200$, whence $x=40\sqrt{\frac{2}{3}}$, which proves to be exactly right.

Ex. 2. Required a rectangle with a perimeter of 100 which shall leave the largest remainder when a square has been taken off one end.

$$\text{Let } x=\text{width, } 50-x=\text{length.}$$
$$50x-x^2=\text{area, } 50x-2x^2=\text{remainder,}$$
$$50x-2x^2=50x-2x^2,$$
$$2(x^2-x^2)=50(x-x)$$
$$2(x+x)=50$$
$$x=12\tfrac{1}{2}.$$

Ex 3. American Mathematical Monthly, No. 3, p. 89.

From a given quantity of material a cylindrical cup with circular bottom and open top is to be made, the cup to contain the greatest amount.

$$\text{Let } m=\text{amount of material,}$$
$$x=\text{radius of bottom,}$$

$$2\pi x = \text{circumference of bottom},$$
$$\pi x^2 = \text{area of bottom},$$
$$m - \pi x^2 = \text{convex area}.$$
$$\frac{m-\pi x^2}{2\pi x} = \text{depth}.$$

Then $\frac{m-\pi x^2}{2\pi x} \times \frac{\pi x^2}{1} = \frac{mx}{2} - \frac{\pi x^3}{2} = \text{capacity}.$

Taking x and y as identical and omitting denominators $mx - \pi x^3 = my - \pi y^3$, or $\pi(x^3 - y^3) = m(x-y)$; whence $\pi(x^2 + xy + y^2) = m$, and $\pi x^2 = \frac{m}{3}$. $\therefore$ the bottom will require $\frac{1}{3}$ of the material.

Then $\frac{2m}{3}$ = convex surface and this divided by circumference of bottom will give the depth.

$\frac{2m}{3} \times \frac{1}{2\pi x} = \frac{m}{3\pi x}$, which by last equation above $= x$.

Hence the depth must equal the radius of the bottom.

Ex. 4. Let the last example be modified by the condition that the cup is made of sheet metal, the bottom being cut from a square piece, and the trimmings wasted. What ratio of depth and diameter will be most economical of material.

Let m = material,
x = radius of bottom,
a = depth.
Then $2\pi x$ = circumference of bottom,
πx^2 = area of bottom,
$4x^2$ = bottom and waste,
$m - 4x^2$ = convex surface,

$\frac{m-4x^2}{2\pi x} = a = \text{depth}$, $\frac{\pi x^2}{1} \times \frac{m-4x^2}{2\pi x} = \frac{mx}{2} - \frac{4x^3}{2} = \text{capacity}$. $mx - 4x^3 = my - 4y^3$, $4(x^3 - y^3) = m(x-y)$, $\therefore 12x^2 = m$.

Substituting this value of m in the value of a above gives $a = \frac{4x}{\pi}$.

$\therefore a : x :: 4 : \pi$.

EDITORIALS.

We have just recently availed ourselves of a set of the Analyst. But the set is not complete as No. 7 Vol. I., No. 6 Vol. V., and No. 1 Vol. IX. are missing. Any of our readers who have these Nos. for sale will do us a favor by writing to us at Kidder, Mo.

Professor J. K. Ellwood should have received credit for solving No. 27, Arithmetic Department.